N79-17478

THEORETIC ASPECTS OF THE IDENTIFICATION OF THE PARAMETERS IN THE OPTIMAL CONTROL MODEL.

Ron A. van Wijk '), Jan J. Kok

Man-Machine Systems Group, Laboratory for Measurement and Control, Department of Mechanical Engineering, Delft University of Technology, Delft, Netherlands.

0. Summary.

The identification of the parameters of the optimal control model (Baron, Kleinman, Levison [1]) from input-output data of the human operator is considered. Accepting the basic structure of the model as a cascade of a full-order observer and a feedback law, and suppressing the inherent optimality of the human controller, the parameters to be identified are the feedback matrix, the observer gain matrix and the intensity matrices of the observation noise and the motor noise.

The identification of the parameters is a statistical problem, because of the fact that the system and output are corrupted by noise, and therefore the solution must be based on the statistics (probability density function) of the input and output data of the human operator. However, based on the statistics of the input-output data of the human operator, no distinction can be made between the observation and the motor noise, which shows that the model suffers from overparametrization. In order to obtain a unique set of parameters, for the model to be identified an equivalent system must be defined, the associated system, in which the observation noise and the motor noise are replaced by an innovations process, which is a combination of these noises.

The parameters in the associated system can be identified if the following conditions are satisfied. First, the input and output signal must be persistently exciting of sufficient order. Second, the parameters of the associated system must satisfy a controllability and an observability condition. This last condition is plausible, for the non-controllable and non-observable part of the system do not influence the output of the system. As a consequence, in an identification procedure only the controllable and observable part of the system can be identified. But even if all the conditions are satisfied, the parameters of the associated system are not uniquely related to the model parameters. In fact the model parameters can be derived from the parameters of the associated system up to a similarity transformation of the model. The optimal control model exhibits a particular structure, due to the cascade of an observer and a feedback law. So,not every similarity transformation on the parameters of the associated system yields a set of parameters corresponding to this structure of the optimal control model. In order to satisfy this requirement the transformation matrix must be a solution of a certain quadratic matrix equation.

Once the parameters are determined, the optimality of the estimated parameters is considered making use of the results of the iverse optimal control problem and its filtering equivalent. The optimality conditions may further restrict the equivalent class of feedback matrices and observer gain matrices. Corresponding to each feedback law a class of weighting matrices exists, just as with the observer gain matrix and the corresponding noise intensity matrices. So the weighting matrices and the observation noise and motor noise cannot be identified uniquely without additional information about these parameters.

') The research reported in this paper is partially supported by the Netherlands Organization for Advancement of Pure Research (ZWO).

1. Introduction.

A very important feature of the optimal control model is the fact that the parameter values which give the best description of the human controller, do satisfy certain general rules, which means that the model in conjunction with these rules for adjustment of its parameters can be used to predict the human controller behavior in a broad class of control situations.

As is known from publications of Kleinman, Baron, Levison [1] successful methods exist for iterative adjustment of the model parameters such that the model agrees adequately with human control behavior. However, this method for the determination of the model parameters also shows some aspects which can be prohibitive for application of the model in investigation of certain questions related to human control behavior. For example, to determine the optimal control law in the model it is assumed that the control task, as performed by the human controller, is known. For those situations where the appointed task is known in advance, and where also care has taken that the human controller optimizes this task,and no other, the given method can be used indeed. However, if one tries to examine the subjective interpretation of the human controller of a certain control task laid upon him, then the optimized performance index is *not* known a priori. In this case one would like to take the reverse approach by determining the control law carried on by the operator from the measurements of his input and output, then verifying the optimality of the control law, and subsequently establishing the corresponding performance index which has been optimized.

Not only in the control part, but also in the observation part of the model such difficulties can arise. According to the rules of parameter adjustment of Baron et. al., the intensity of the observation noise has a certain fixed connection with the power density of the observer signal. This statement is based on the assumption that the display used for observation of the system output is a standard design which fulfills standard norms. However, if one considers the question of the influence of display parameters on the corresponding observation noise, then again one is confronted with the reverse problem of determining the intensity of the observation noise directly from the measurements. Additional to this problem at the observation side,the motor side shows a similar problem related to the controls and the motor noise.

The problems mentioned above have motivated us to concentrate the investigation of the optimal control model on the development of an identification procedure, which allows for a direct estimation of the parameters of the model from measurements of the input and output of the human controller.

It should be mentioned that with respect to the general optimal control model as introduced by Baron, Kleinman and Levison,a simplification will be

introduced in the form of the suppression of the inherent time delay of the human controller and, consequently, of the predictor in the model. Such a simplification might be inadmissible for the application of the model in relatively fast vehicle control situations (like automobile and aircraft control), where the inherent time delay plays an important role, but seems to be acceptable for application of the model in relatively slow control situations; particularly the application of this simplified model in supervisory control.

2. *Structure of the model and the model equations.*

Referring to Fig 1 which gives the structure of the model, the closed loop system consisting of the controlled system and the human operator model, is described by the following set of equations:

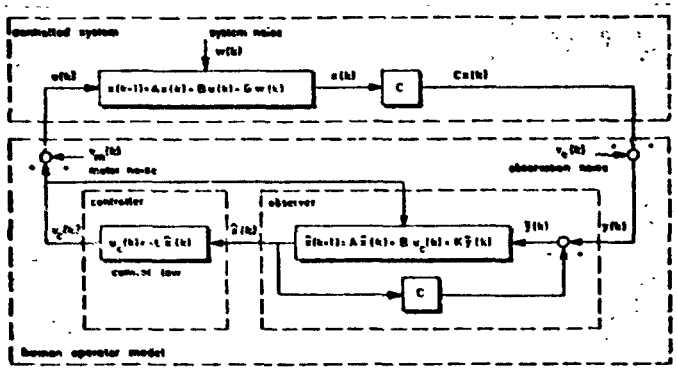

Fig. 1: Structure of the human operator model.

- State equation of the controlled system

$$x(k+1) = Ax(k)+Bu(k)+Gw(k) \; ; \tag{2.1}$$

- observation equation

$$y(k) = Cx(k)+v_o(k) \; ; \tag{2.2}$$

- state equation of the observer

$$\hat{x}(k+1) = A\hat{x}(k)+Bu_c(k)+K(y(k)-C\hat{x}(k)) \; ; \tag{2.3}$$

- control law

$$u_c(k) = -L\hat{x}(k) \; ; \tag{2.4}$$

- control equation

$$u(k) = u_c(k)+v_m(k) \; . \tag{2.5}$$

The various quantities involved are defined as follows:

$x(k)$ = state of the controlled system (n- dimensional);
$u(k)$ = input of the controlled system (m- dimensional);
$w(k)$ = system noise (stationary d_w- dimensional white noise with intensity matrix ϕ_w);
$v_o(k)$ = observation noise (stationary d_y- dimensional white noise with intensity matrix ϕ_{v_o});
$y(k)$ = observed output variable (d_y- dimensional);
$\hat{x}(k)$ = state of the observer (the one step predictor estimate $\hat{x}(k/k-1)$);
$u_c(k)$ = commanded input;
$v_m^c(k)$ = motor noise (stationary m- dimensional white noise with intensity matrix ϕ_{v_m});
A, B, C, G = a priori known system matrices;
L = feedback matrix;
K = observer gain matrix.

The noises $w(k)$, $v_o(k)$ and $v_m(k)$ are assumed to be mutually uncorrelated.

If L is equal to the optimal feedback matrix L^o which optimizes the performance index:

$$J = \lim_{K\to\infty} \frac{1}{K} E\{ \sum_{k=k_o}^{K} x^T(k)Qx(k)+u^T(k)Ru(k)\} \; , \tag{2.6}$$

then:

$$L^o = R^{-1}B^TA^{-T}(P-Q) = (R+B^TPB)^{-1}B^TPA \; , \tag{2.7}$$

where the matrix P is the unique symmetric positive definite solution of the algebraic Riccati equation:

$$P = Q+A^TP[I+BR^{-1}B^TP]^{-1}A = Q+A^TPA-A^TPB[B^TPB+R]^{-1}B^TPA \; . \tag{2.8}$$

The weighting matrices Q and R in the performance index (2.6) are assumed to be symmetric and (semi) positive definite.

If K is equal to the optimal observer gain matrix K^o which minimizes the mean quadratic reconstruction error (the matrix W is a symmetric positive definite matrix):

$$J = E\{\tilde{x}^T(k)W\tilde{x}(k)\} \; , \tag{2.9}$$

where $\tilde{x}(k)$ is the reconstruction error:

$$\tilde{x}(k) = x(k)-\hat{x}(k) \; , \tag{2.10}$$

then:

$$K^o = AV_{\tilde{x}}C^T[\phi_{v_o}+CV_{\tilde{x}}C^T]^{-1} = [V_{\tilde{x}}-G\phi_wG^T-B\phi_{v_m}B^T]A^{-T}C^T\phi_{v_o}^{-1} \; , \tag{2.11}$$

where the matrix $V_{\tilde{x}}$, the covariance matrix of the reconstruction error, is the unique symmetric positive definite solution of the algebraic Riccati equation:

$$V_{\tilde{x}} = B\phi_{v_m}B^T+G\phi_wG^T+AV_{\tilde{x}}[I+C^T\phi_{v_o}^{-1}CV_{\tilde{x}}]^{-1}A^T =$$

$$B\phi_{v_m}B^T+G\phi_wG^T+AV_{\tilde{x}}A^T-AV_{\tilde{x}}C^T[CV_{\tilde{x}}C^T+\phi_{v_o}]^{-1}CV_{\tilde{x}}A^T \; . \tag{2.12}$$

The intensity matrices ϕ_w, ϕ_{v_m}, ϕ_{v_o} are assumed to be symmetric and (semi) positive definite.

In the case that the performance index also includes the control rate the model remains of the form (2.3) - (2.5) by proper augmentation of the state vector.

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3. *The choice of model parameters in relation to the identifiability.*

Having fixed the overall structure of the model, the parameters of the model will be considered now. Following the line of the optimal control model of Baron, Kleinman and Levison the parameter set consists of the set of matrices $\{Q,R,\psi_{v_m},\psi_{v_o}\}$, so the parameters of the performance index and of the statistics of the noises. In that case the feedback matrix L^o and the observer gain matrix K^o are considered as dependent parameters (L^o is the optimal feedback matrix corresponding to the weighting matrices Q and R, and K^o is the optimal observer gain matrix corresponding to the intensity matrices ψ_{v_m} and ψ_{v_o}). However, if the optimality of the human controller is not a priori accepted, then the parameter set consists of the matrices $\{L,K,\psi_{v_m},\psi_{v_o}\}$. The questions whether a given feedback matrix L is indeed optimal in the sense of some quadratic performance index and, if so, what is the corresponding set of weighting matrices Q and R, are then considered separately. The same counts for the optimality of a given observer gain matrix K and the corresponding intensity matrices ψ_{v_m} and ψ_{v_o}. This way of approach has some important advantages in the identification of the model from input-output data, like (possibly) a better fit of the data (a broader class of matrices L and K is allowed) and less problems with the uniqueness of the solution (a particular matrix L or K corresponds with a whole class of possible matrices Q and R or matrices ψ_{v_m} and ψ_{v_o}. The test whether the matrices L and K are solutions of an optimal control problem or observer problem respectively, can be verified by application of the results of the inverse optimal control problem and observer problem respectively (Sec. 9). This approach means basically that the structure of the optimal control model is fully accepted and adopted, however, that the optimality of the human controller is not assimilated in advance, and that it will be examined afterwards.

Based on the considerations given above the identification problem of the human operator model is defined in terms of the parameter set $\{L,K,\psi_{v_m},\psi_{v_o}\}$. In the evaluation of the identifiability of this parameter set it is assumed that there are experimental measurements available of the closed loop system (Fig 1), particularly of the control signal u(k) and the output signal Cx(k).

The system equation and the output equation of the system to be identified (2.3) - (2.5) can be written in the following form:

$$\hat{x}(k+1) = (A-KC-BL)\hat{x}(k)+KCx(k)+Kv_o(k) \; ; \qquad (3.1)$$

$$u(k) = -L\hat{x}(k)+v_m(k) \; . \qquad (3.2)$$

In the system (3.1), (3.2), the input Cx(k) and the output u(k) are measured and the system matrices A,B,C and G are assumed to be known.

In the following sections the problem will be treated whether the parameters L,K,ψ_{v_m} and ψ_{v_o} of the system (3.1), (3.2) operating in a closed loop are identifiable and under what conditions. First some general results of the identifiability concept will be presented, which then will be applied to the underlying problem.

4. *General results of the identifiability concept.*

Before solving the parameter identification problem, the problem of the identifiability of the parameters should be considered. In case of the model structure is given, the question also arises whether the input signals are appropriate for the identification of the unknown parameters in the system. The identifiability problem can be approached in a deterministic way or a stochastic way. Tse and Anton [2] argue that deterministic identifiability is necessary but not sufficient for stochastic identifiability when the output and (or) the system is corrupted by noise. This is plausible, for in a deterministic way nothing can be said about the statistics of the noise sources. So, in the given stochastic case the system parameters have to be recovered in some probabilistic sense.

For the elaboration of the identifiability we introduce the following notation:
The observations u(k) will be denoted by z(k); the set of observations z(1) for $k_o \le l \le k$ will be written as Z(k):

$$Z(k) \triangleq \{z(k_o), \ldots, z(k)\} \; ; \qquad (4.1)$$

the vector of unknown parameters will be denoted by θ, in the underlying problem $\theta = \{K,L,\psi_{v_m},\psi_{v_o}\}$; (4.2)
the probabilitydensity function of z(k) conditioned on Z(k-1) and θ is: $p(z(k)/Z(k-1),\theta)$.

Based on statistical considerations, Tse and Anton [2] defined resolvability of two parameters in the following way:

> Two parameters θ_1 and θ_2 belonging to some compact (s- dimensional) set R_c^s are *unresolvable* if the equality:
>
> $$p(z(k)/Z(k-1),\theta_1) = p(z(k)/Z(k-1),\theta_2) \qquad (4.3)$$
>
> holds with probability 1 for all, except a finite number of integers $k \ge k_o$.

Under some mild conditions resolvability guarantees the existence of a consistent sequence of estimates, which means the identifiability of parameters in a probabilistic sence. One of these conditions is the requirement that the observation statistics z(k) are asymptotically uncorrelated. If the closed loop system is stable, which will be the case in a normal control situation, this condition is always satisfied. It easily can be derived that in that case A-BL and A-KC are stable matrices.

So, in this section the conclusion is drawn that the identifiability of the parameters have to be based on the probability density function of the measurements. Tse and Weinert [3] have applied this concept to a linear system without an additional input signal. In the following section the results will be applied to the problem of the identifiability of the parameters in the system given by (3.1) and (3.2), which is more difficult because of the extra input Cx(k).

5. *Application of the identifiability results to the linear system.*

The closed loop system (2.1) - (2.5) is linear and the noise sources are assumed to be Gaussian. In this case the probability density function $p(z(k)/Z(k-1),\theta)$ is completely characterized by its mean and variance:

$$\hat{z}(k/k-1,\theta) \triangleq E\{z(k)/Z(k-1),\theta\} \; ; \qquad (5.1)$$

$$V_z(k/k-1,\theta) \triangleq E\{(z(k)-\hat{z}(k/k-1,\theta))(z(k)-\hat{z}(k/k-1,\theta))^T/Z(k-1),\theta\}. \qquad (5.2)$$

In the stationary control situation the variance has a constant value which will be denoted by V_z. According to Eq.(4.3), the identifiability of the parameters now can be related to $\hat{z}(k/k-1,\theta)$, and to V_z. So, two parameters θ_1 and θ_2 are resolvable if:

$$\hat{z}(k/k-1,\theta_1) \neq \hat{z}(k/k-1,\theta_2) \; ; \tag{5.3}$$

$$V_z(\theta_1) \neq V_z(\theta_2) \; . \tag{5.4}$$

for all except a finite number of integers $k \geq k_o$.

For the linear system to be identified with Gaussian inputs :

$$\xi(k) = A^*\xi(k)+Kcx(k)+Kv_o(k) \; ; \tag{3.1}$$

$$z(k) \triangleq u(k) = -L\xi(k)+v_m(k) \; ; \tag{3.2}$$

$$A^* \triangleq A-KC-BL \; , \tag{5.5}$$

the mean (5.3) and variance (5.4) can be calculated. A Kalman filter (one stage predictor) of the system (3.1), (3.2) generates a linear projection of the state $\hat{x}(k)$ on the measurement space $Z(k-1)$ which for the linear Gaussian case is equal to:

$$\xi(k) \triangleq E\{\hat{x}(k)/Z(k-1),\theta\} \; . \tag{5.6}$$

So a possible way to generate $\hat{z}(k/k-1,\theta)$ is:

$$\hat{z}(k) \triangleq \hat{z}(k/k-1,\theta) = -L\xi(k) \; . \tag{5.7}$$

Here $\xi(k)$ is the output of the Kalman filter:

$$\xi(k+1) = A^*\xi(k)+Kcx(k)+Fv(k) \; ; \tag{5.8}$$

$$z(k) = -L\xi(k)+v(k) \; . \tag{5.9}$$

In this algorithm the observed signal $Cx(k)$ represents a known input to the filter, $v(k)$ is the innovations process, which is a white noise process with intensity ψ_v, and the matrix F is the filter gain matrix. The variance V_z results from the Eqs. (5.2), (5.7) and (5.9):

$$V_z = \psi_v \; . \tag{5.10}$$

As mentioned before the algorithm (5.7) - (5.9) is just one possible way to obtain $\hat{z}(k)$. It is clear by direct substitution that any system related to (5.7) - (5.9) in the following way (for every non-singular T):

$$\hat{A}^* = TA^*T^{-1}; \; \hat{K} = TK; \; \hat{F} = TF; \; \hat{L} = LT^{-1} \tag{5.11}$$

generates the same $\hat{z}(k)$ and V_z. Thus,the system with which $\hat{z}(k)$ and V_z can be calculated will be written by:

$$\xi(k+1) = \hat{A}^*\xi(k)+\hat{K}Cx(k)+\hat{F}v(k) \; ; \tag{5.12}$$

$$z(k) = -\hat{L}\xi(k)+v(k) \; ; \tag{5.13}$$

$$\hat{z}(k) = -\hat{L}\xi(k) \; . \tag{5.14}$$

As a first result ,it can be noticed that in consequence of the statistical approach it is not possible, from measurements $Cx(k)$ and $z(k)$ only, to distinguish between the system (5.12), (5.13) and the system to be identified (3.1), (3.2). In the system (5.12), (5.13), which is called the *associated system*, the observation noise and the motor noise are replaced by the equivalent white noise source $v(k)$ and the gain matrix $\hat{F}$. So the conclusion can be drawn that, from the measurements $Cx(k)$ and $z(k)$, it is *not* possible to determine each of the intensities of the observation noise ψ_{v_o} and of the motor noise ψ_{v_m},seperately. Only the equivalent white noise source $v(k)$ and the gain matrix can be identified instead. This conclusion implies that the parameter set to be identified has to be adjusted to the parameters of the associated system:

$$\hat{\theta} = \{\hat{A}^*, \hat{K}, \hat{L}, \hat{F}, \hat{\psi}_v\} \; . \tag{5.15}$$

On behalf of the application of the identifiability results (5.3), (5.4) to the associated system, this system will be rewritten in the following form:

$$\xi(k+1) = (\hat{A}^*+\hat{F}\hat{L})\xi(k)+[\hat{K}|\hat{F}]\gamma(k) \; ; \tag{5.16}$$

$$\hat{z}(k) = -\hat{L}\xi(k) \; . \tag{5.17}$$

Here the input signal $\gamma(k)$ is a composition of the measured input and output of the system to be identified, and it is completely known:

$$\gamma(k) = \begin{bmatrix} Cx(k) \\ z(k) \end{bmatrix} \; . \tag{5.18}$$

Since the closed loop system is assumed to be stable, also $(\hat{A}^*+\hat{F}\hat{L})$ is a stable matrix, so for the stationary situation the response of the system (5.16), (5.17) can be written as:

$$\hat{z}(k) = \sum_{i=0}^{k-k_o-1} -\hat{L}(\hat{A}^*+\hat{F}\hat{L})^i[\hat{K}|\hat{F}]\gamma(k-i-1) \; , \tag{5.19}$$

or:

$$\hat{z}(k) = \sum_{i=0}^{k-k_o-1} H(i/\hat{\theta})\gamma(k-i-1) \; , \tag{5.20}$$

where theimpulse response matrix $H(i/\hat{\theta})$ is defined by:

$$H(i/\hat{\theta}) = -\hat{L}(\hat{A}^*+\hat{F}\hat{L})^i[\hat{K}|\hat{F}] \; . \tag{5.21}$$

From the identifiability condition (5.3) and Eq (5.20) it follows that if two different impulse response matrices $H(i/\theta_1)$ and $H(i/\theta_2),(i\geq 0)$, with the same input $\gamma(k)$ give the same response $\hat{z}(k)$ the parameters θ_1 and θ_2 are unresolvable. Thus from condition (5.3) the conclusion can be drawn that for a given input $\gamma(k)$ and given output $\hat{z}(k)$ the impulse response matrix $H(i/\hat{\theta})$, $i\geq 0$, has to be unique. In the next section this will be worked out in detail. Then also the question should be considered under what conditions the parameters , from the impulse response matrix, can be resolved unicuely. This problem will be solved in Sec.7.

From the identifiability condition (5.4), and the Eqs. (5.13) and (5.14), the uniqueness of ψ_v follows as a result.

6. *Uniqueness of the impulse response matrix.*

We will now examine under what conditions the impulseresponse matrix $H(i/\hat{\theta})$, $i\geq 0$, is unique for a given input $\gamma(k)$ and a unique output signal $\hat{z}(k)$, $k\geq k_o$. Since the parameters to be identified are determined by the impulse responsematrix (5.21), the side constraint has to be posed that

$H(i/\hat{\theta})$, $i>0$, is a realizable impulse response matrix of a finite dimensional time invariant system. In this case it follows from the Cayley-Hamilton theorem that a matrix $(\hat{A}^*+\hat{F}\hat{L})^i$, $i\geq n$, is a linear function of the matrices $(\hat{A}^*+\hat{F}\hat{L})^i$, $i=0,\ldots,n-1$. Thus, it is only necessary to work out the conditions for the set $H(i/\hat{\theta})$, $i=0,\ldots,n-1$, to be unique.

Suppose now that two impulse response matrices $H_1(i/\hat{\theta})$ and $H_2(i/\hat{\theta})$ satisfy the Eq.(5.20). Subtraction of both the equations leads to the result:

$$0 = \sum_{i=0}^{k-k_o-1} (H_1(i/\hat{\theta})-H_2(i/\hat{\theta}))\gamma(k-i-1) \quad . \tag{6.1}$$

From Eq (6.1) it can be concluded that:

$$[\Delta H(0)\ldots\Delta H(k-k_o-1)]\begin{bmatrix} E\{\gamma(k-1)\gamma^T(k-1)\}\ldots E\{\gamma(k-1)\gamma^T(k_o)\} \\ \vdots \qquad\qquad \vdots \\ E\{\gamma(k_o)\gamma^T(k-1)\}\ldots E\{\gamma(k_o)\gamma^T(k_o)\} \end{bmatrix}\begin{bmatrix} \Delta H^T(0) \\ \vdots \\ \Delta H^T(k-k_o-1) \end{bmatrix} = 0 \; ; \tag{6.2}$$

with:

$$\Delta H(j) = H_1(j/\hat{\theta})-H_2(j/\hat{\theta}) \quad . \tag{6.3}$$

Because of the fact that only the uniqueness of $H(i/\hat{\theta})$, $i=0,\ldots,n-1$ must be shown, $k = k_o+n$ will be choosen. Then, if:

$$\begin{bmatrix} E\{\gamma(k_o+n-1)\gamma^T(k_o+n-1)\}\ldots E\{\gamma(k_o+n-1)\gamma^T(k_o)\} \\ \vdots \qquad\qquad \vdots \\ E\{\gamma(k_o)\gamma^T(k_o+n-1)\} \quad \ldots E\{\gamma(k_o)\gamma^T(k_o)\} \end{bmatrix} > 0 \; ; \tag{6.4}$$

(>0 means: the matrix is positive definite) from Eq (6.2) it follows that:

$$[\Delta H(0)\ldots\Delta H(k-k_o-1)] = 0 \quad , \; k = k_o+n \quad . \tag{6.5}$$

From Eq.(6.5), by using Eq.(6.3) and the fact that $H(i/\hat{\theta})$, $i\geq n$, is a linear function of $H(i/\hat{\theta})$, $i=0,\ldots,n-1$, it results that the impulse response matrix $H(i/\hat{\theta})$ is unique. A signal $\gamma(k)$ with the property given by the Eq. (6.4) is called *persistently exciting of order (n-1)*. A sufficient condition for this property can also be given in the (discrete) frequency domain. The signal $\gamma(k)$ is persistently exciting of infinite order if:

$$S_{\gamma\gamma}(z) > 0, \quad z = e^{j\omega}; \quad 0 < \omega \leq 2\pi \quad , \tag{6.6}$$

where $S_{\gamma\gamma}(z)$ is the discrete spectral density matrix of $\gamma(k)$. From the condition (6.6) it is easily seen that if $\gamma(k)$ is a white noise source, it is persistently exciting of infinite order. This was the case Tse and Weinert [3] have considered. In our case it is not possible, only based on the structure of the system to be identified, to show directly that $\gamma(k)$ is persistently exciting of sufficient order. But, in the given situation the signals $r(k)$ and $Cx(k)$, and thus $\gamma(k)$, are known, so it is possible to test the condition (6.4). If this condition is satisfied, the impulse response matrix $H(i/\hat{\theta})$, $i=0,1,\ldots$ turns out to be unique. However, the conditions (6.4) and (6.6) are only sufficient, so if (6.4) (or (6.6)) is not satisfied, it does not imply that the impulse response matrix $H(i/\hat{\theta})$ is not unique.

7. *Realization of the impulse response matrix.*

After application of the test (6.4) for the measured signals the next step in the identifiability problem is to work out the uniqueness and the structure of the parameters given the uniqueness of the impulse response matrix. This problem is known in literature as the *realization problem*. An extensive treatment of this problem is given by Silverman [4].

The following definitions will be used:

Definition 7.1:
A system $(A,B,C)_n$ *) is *H-equivalent* to $(\bar{A},\bar{B},\bar{C})_n$ if and only if the impulse response matrices of both the systems are equal.

Definition 7.2:
A system $(A,B,C)_n$ is *strictly equivalent* to $(\bar{A},\bar{B},\bar{C})_n$ if and only if there exists a constant non-singular matrix T such that:

$$\bar{A} = TAT^{-1}; \; \bar{B} = TB; \; \bar{C} = CT^{-1} \quad . \tag{7.1}$$

Definition 7.3:
The system $(A,B,C)_n$ with dimension n is called *minimal* if no H-equivalent system $(\bar{A},\bar{B},\bar{C})_{\bar{n}}$ exists with dimension $\bar{n} < n$.

Using the concepts given above, Silverman proved the following theorem.

Theorem 7.1:
Minimal time-invariant system representations are strictly equivalent if and only if they are H-equivalent.

Applied to the situation in which the impulse response matrix $H(i/\hat{\theta})$ is given by Eq.(5.21), this theorem gives the result that *every* minimal H-equivalent system of $(\hat{A}^*+\hat{F}\hat{L},[\hat{K}|\hat{F}],-\hat{L})_n$ is given by:

$$\bar{\hat{A}}^*+\bar{\hat{F}}\bar{\hat{L}} = T(\hat{A}^*+\hat{F}\hat{L})T^{-1}; \; [\bar{\hat{K}}|\bar{\hat{F}}] = T[\hat{K}|\hat{F}]; \; -\bar{\hat{L}} = -\hat{L}T^{-1} \quad . \tag{7.2}$$

In Sec.5 (eqs.(5.8) and (5.9)) it is argued that the system $(A^*+FL,[K|F],-L)_n$ is one of the strictly equivalent systems, so statement (7.2) can also be put in the following form (after proper adjustment of the transformation matrix T):

Every minimal H-equivalent system of $(A^*+FL,[K|F],-L)_n$ is given by:

$$\bar{\hat{A}}^*+\bar{\hat{F}}\bar{\hat{L}} = T(\hat{A}^*+\hat{F}\hat{L})T^{-1}; [\bar{\hat{K}}|\bar{\hat{F}}] = T[\hat{K}|\hat{F}]; \; -\bar{\hat{L}} = -\hat{L}T^{-1}. \tag{7.3}$$

The above statement is equivalent with:

$$\bar{\hat{A}}^* = T\hat{A}^*T^{-1}; \; \bar{\hat{K}} = T\hat{K}; \; \bar{\hat{F}} = T\hat{F}; \; \bar{\hat{L}} = \hat{L}T^{-1}. \tag{7.4}$$

From the total class of possible parameters (7.4) a unique set must be selected. This is considered in the following theorem.

Theorem 7.2:
The matrices $\hat{A}^*,\hat{K},\hat{F},\hat{L}$ can be uniquely identified from the unique impulse response matrix $H(i/\hat{\theta})$ if and only if:

$$(\hat{A}^*,\hat{L}) \text{ is a reconstructable pair,} \tag{7.5}$$

*) The notation $(A,B,C)_n$ represents a linear time invariant n-dimensional system with system matrix A, input matrix B and output matrix C.

- $(\hat{A}^*,[K|F])$ is a controllable pair, (7.6)
- A canonical form for the system has been chosen. (7.7)

No strict proof of this theorem will be given. The conditions are made plausible in the following way:
If a matrix pair $(\hat{A}^*,\hat{L})$ is not reconstructable,then, according to Popov [5], the pair $(\hat{A}^*+\hat{F}\hat{L},\hat{L})$ is also not reconstructable. So, the not-reconstructable subsystem does not influence the output and, thus is not represented in the impulse response matrix. As a consequence, the non-reconstructable subsystem cannot be identified from the impulse response matrix. A similar argument holds for the controllability condition.
All minimal H-equivalent systems are given by Eq.(7.4), where T is some constant non-singular matrix. One way to guarantee the uniqueness of the parameter set is to choose a transformation matrix T such that a canonical parameter set results. For a proper canonical form only a unique transformation matrix T exists. The structure of the canonical form is determined by the so called *Kronecker invariants* (i.e. Popov [6]). The Kronecker invariants are given by the observability matrix of the system and remain unchanged under a similarity transformation. The consequence for the identification procedure is that prior to a parameter identification a *structural identification* must be carried out.

So far,no use has been made of the fact that the parameters of the controlled system (the matrices A, B, C and G) are given in advance, and thus that there exists a certain relation with the parameters to be identified. In particular:

$$\hat{A}^* = \widehat{(A-KC-EL)} = A-\hat{K}C-B\hat{L} \ . \qquad (7.8)$$

Then, by Eq.(7.4), the following quadratic equation results:

$$T\hat{B}\hat{L}T+\hat{A}^*T-TA+\hat{K}C = 0 \ . \qquad (7.9)$$

So,to obtain a set of parameters which satisfies the given structure of the model, the transformation matrix T must be an element of the finite set of solutions of the quadratic equation (7.9). Moreover, from observability considerations, it can be shown that this transformation matrix must be real. In Potter [7] the solution of a more specific quadratic equation is given, which can be generalized to the underlying equation. The maximum number of real regular solutions T which satisfy Eq.(7.9) is equal to $\binom{2n}{n}$. Using the controllability condition (7.5), it can be shown that to each real regular solution T one and only one triple $\hat{K}$, $\hat{F}$ and $\hat{L}$ corresponds. Thus, there are as many solutions of the realization problem as there are solutions T of Eq.(7.9), but in general no system in canonical form will belong to the set of possible solutions. A way to work out the identification is to identify the structure of the system, then to choose a canonical form,, to identify the parameters in the accepted canonical form, and to transform the canonical parameters using Eq.(7.9) to the ultimate parameter set. It should be noticed that the set of solutions is not influenced by the particular choice of the canonical form (with proper invariants).

In the introduction (Sec.1) it was argued that, once the parameters K and L are identified, the optimality of these parameters should be verified which might restrict the total set of possible solutions. In the following section some results will be presented in relation to this problem.

8. The optimality of the feedback matrix and the observer gain matrix.

So far, no assumption was made relative to the optimality of the feedback matrix L and the observer gain matrix K. In this section some results of the so called inverse optimal control and observer problem, which is considered in more detail in van Wijk, Kok [9], will be applied to the identified parameters L and K. The main part of the inverse optimal control problem concerns the question under which conditions a given feedback matrix L, in relation to the system matrices A and B, minimizes a quadratic cost function (2.6).
A similar question holds for the inverse observer problem, i.e. under which conditions a given gain matrix K, in relation to the matrices A and C, minimizes the quadratic reconstruction error (2.9).

For the control part it can be shown that if the matrix:

$$LA^{-1}B \text{ is } \textit{symmetrizable} \ , \qquad (9.1)$$

the feedback matrix L minimizes some quadratic criterion of the form (2.6). The concept of symmetrizability is defined by Taussky [10] in the following way:

> A matrix is symmetrizable if it is similar to a symmetric matrix, or, equivalently, if it has real characteristic roots and a full set of characteristic vectors.

The symmetrizability requirement is plausible indeed, for an optimal feedback matrix L^o does satisfy Eq.(2.7), so:

$$RL^o = B^TA^{-T}(P-Q) \ . \qquad (9.2)$$

With the symmetry of P and Q it follows that $RL^oA^{-1}B$ is a symmetric matrix. From the fact that R is positive definite it follows that $L^oA^{-1}B$ is similar to a symmetric matrix, hence the matrix is symmetrizable.

If the symmetrizability condition is satisfied, only the positive definiteness of the weighting R of the input is guaranteed, but the weighting matrix Q of the state is not necessarily semi positive definite. If, in addition, Q is required to be semi positive definite, also the following two necessary conditions must be satisfied by L:

- The *rank condition:*

$$r\{L\} = r\{LA^{-1}B\} \ ; \qquad (9.3)$$

- The *eigenvalue condition:*

$$0 \le \lambda_i\{LA^{-1}B\} < 1 \ . \qquad (9.4)$$

The rank condition follows from the fact that the Riccati matrix P is also a semi positive definite matrix, and from the following relation, derived from the Eqs.(2.7) and (2.8):

$$(R+B^TPB)L^oA^{-1}B = B^TPB \ . \qquad (9.5)$$

The eigenvalue condition can be shown along the semi positive definiteness of P, and another relation resulting from the Eqs.(2.7) and (2.8):

$$F(I-L^oA^{-1}B)^{-1}L^oA^{-1}B = B^TPB \ge 0 \ . \qquad (9.6)$$

For a given feedback matrix which is obtained by identification, the above conditions can be tested in order to examine its optimality. In [9] the complete class of weighting matrices Q and R belonging to L^o is deduced. From this, the conclusion can be drawn that without additional structural information about the weighting factors no unique pair R and Q can result.

Based on the duality between the inverse optimal control and observer problem the following conditions can be concluded for the optimality of the observer gain matrix K

- The *symmetrizability condition:*

  The matrix $CA^{-1}K$ is symmetrizable. (9.7)

- The *rank condition:*

  $$r\{K\} = r\{CA^{-1}K\} \quad . \tag{9.8}$$

- The *eigenvalue condition:*

  $$0 \leq \lambda_i\{CA^{-1}K\} < 1 \quad . \tag{9.9}$$

Once a matrix K is identified, the conditions given above can be used to check the optimality of the observer.

In Sec.8 it was concluded that no unique pair {L,K} can be the solution of the identification problem, but it was derived that:

$$\hat{L} = LT^{-1}, \quad \hat{K} = TK \tag{9.10}$$

The transformation matrix T is a real regular solution of the quadratic equation (7.9). The given optimality conditions must be checked for every pair $\{\hat{L},\hat{K}\}$. Because of the fact that the optimality conditions are *not* invariant under the transformation (9.10), these conditions may restrict the total class of possible solutions. However, to each pair {L,K} for which the optimality test is successful, a whole class of weighting matrices in the performance index (2.6) corresponds. The same is true for a whole class of intensity matrices of the observation noise and motor noise, respectively. So, without additional information, no unique model parameter values Q, R, ψ_{v_o}, ψ_{v_m} can be identified from the input-output data of the human operator.

9. *Discussion of the identification and optimality conditions.*

The identifiability of the parameters was based on the probability density function of the measurements. In consequence of the statistical considerations, no distinction could be made between the system to be identified, Eqs.(3.1) and (3.2), and the associated system, Eqs.(5.12) and (5.13). It was argued that it is not possible to identify the observation noise and motor noise as independent noise sources, but instead, the intensity of an equivalent white noise innovations process v(k) and its input matrix F can be determined only. For the identification of the parameters a procedure can be used as described in Kok, van Wijk [8]. The parameter ψ_v can always be identified uniquely; the other parameters of the associated system can be identified under the following (sufficient) conditions:

a) The signal $\begin{bmatrix} x(k) \\ r(k) \end{bmatrix}$ is persistently exciting of order (n+1),

b) $(\hat{A}^*, \hat{K}|\hat{F}|)$ is a controllable pair,

c) $(\hat{A}^*, \hat{L})$ is a reconstructable pair.

d) A canonical form has been chosen for $(\hat{A}^*, [\hat{K}|\hat{F}], \hat{L})$.

Condition a) can be tested prior to the identification. However, if this condition is not satisfied it does not necessarily mean that the parameters cannot be identified at all.
It is not possible to test conditions b) and c) in advance. However, the controllability and the reconstructability must be considered as inherent properties of the results of an identification procedure.
In relation to condition d) it was argued that in general the parameters of the model to be identified will *not* represent a canonical structure. However, in the application of an identification procedure it still is possible to make use of an a priori determined canonical form with proper Kronecker invariants. Once the parameters of the model in canonical form are identified, they must be transferred to a set of parameters which fits the subsumed model structure. The quadratic equation (7.9) determines the set of possible (real regular) transformation matrices T which will provide for this conversion. In general more than one (maximal $\binom{n}{n_1}$) real regular solution T will exist. However, if the quadratic equation has *not* at least one real regular solution T, it turns out that *no* set of parameters with the subsumed structure in the model fits the input-ouput data. In this case the assumptions of the model structure must be abandoned.

In the above, the structure of the system to be identified (i.e. the Kronecker invariants) is assumed to be known in order to be able to identify the parameters. So, prior to the parameter identification, a structural identification must be worked out. In literature however, very few procedures are available for this purpose.

In conclusion it can be stated that in general more than one set of the parameters $\{K,L,\psi_v,F\}$ is the possible solution of the identification problem, and therefore some additional information should be available in order to determine a unique parameter set. Once the parameters L and K are determined, the optimality of these parameters can be checked using conditions (9.1), (9.3), (9.4) and (9.7) - (9.9), respectively.

As an overall conclusion it can be stated that, from the identification point of view, the optimal control model suffers from overparametrization. The observation noise and motor noise are not resolvable, and the feedback matrix and the observer gain matrix can only be determined up to a similarity transformation of the model, where the transformation matrix is restricted to the solution of a quadratic equation.

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